IX. *On the General Refolution of Algebraical Equations. by* Edward Waring, *M. D. F. R. S. and of the Inftitute of* Bononia, *Lucafian Profeffor of Mathematics in the Univerfity of* Cambridge.

Read Jan. 28, 1779.

IN the year 1757 I fent fome papers to the Royal Society, which papers were printed in the year 1759, and copies of them delivered to feveral perfons; thefe papers fomewhat corrected, with the addition of a fecond part on the properties of curve lines, were publifhed in the year 1762. In the years 1767, 1768 and 1769 I printed, and publifhed in the beginning of the year 1770, the fame papers with additions and emendations under the title of Meditationes Algebraicæ. In thefe papers were contained, with many other inventions, the moft general refolution of algebraical equations known, as it contains the refolution of every algebraical equation, of which the general refolution has been given, viz. the refolution of quadratic, cubic and biquadratic, the refolution of Mr. DE MOIVRE's and Mr. BEROUT's (fince publifhed) equations; it difcovers the refolution

refolution of an equation of n dimenfions, of which the n roots are given, and alfo deduces innumerable equations of n dimenfions, which contain $n-1$ independent coefficients. From whence it feems probable, that this new method of mine may contain the moft general refolution of algebraical equations that ever has, or, perhaps, ever will be invented.

The general refolution is $x = a\sqrt[n]{p} + b\sqrt[n]{p^2} + c\sqrt[n]{p^3} + d\sqrt[n]{p^4} \ldots + r\sqrt[n]{p^{n-3}} + s\sqrt[n]{p^{n-2}} + t\sqrt[n]{p^{n-1}} + \frac{A}{n}$, if the equation be $x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + Dx^{n-4} -$ &c. $= 0$.

I fhall add the refolution of fome particular equations from this method, and then fubjoin the equation to which $x = a\sqrt[n]{p} + b\sqrt[n]{p^2} + c\sqrt[n]{p^3} +$ &c. is the general refolution.

1. Let the refolution be $x = a\sqrt[3]{p} + b\sqrt[3]{p^2}$, and the correfpondent equation free from radicals will be found $x^3 - 3abpx - a^3p - b^3p^2 = 0$. Let $x^3 - Px - Q = 0$ be a cubic equation whofe refolution is required, which fuppofe the fame as the equation found above, and confequently their correfpondent terms equal, i. e. $P = 3abp$ and $Q = a^3p + b^3p^2$, whence $p = \frac{P}{3ab}$, which value being fubftituted for p in the fecond equation, there refults $Q = \frac{Pa^2}{3b} + \frac{bP^2}{9a^2}$. In this equation for a or b may be affumed unity, or any other

other quantity whatever, and there will refult an equation of the formula of a quadratic from which the other b or a may be found, whence from the equation ($p = \frac{P}{3ab}$) p may be deduced, and confequently the refolution of the cubic required.

In the fame manner for p may be affumed any quantity whatever, and in the equation $Q = a^3p + b^3p^2$ for b fubftitute its value $\frac{P}{3ap}$, or for a its value $\frac{P}{3bp}$, and there refult the equations $Q = a^3p + \frac{P^3}{27a^3p}$, and $Q = \frac{P^3}{27b^3p^2} + b^3p^2$, which have the formula of a quadratic, from which may be deduced the refolution of the cubic required.

2. Let the refolution affumed be $x = a\sqrt[4]{p} + b\sqrt[4]{p^2} + c\sqrt[4]{p^3}$; exterminate the irrational quantities, and there refults the equation $x^4 - (2b^2 + 4ac)px^2 - 4(a^2bp + bc^2p^2)x - a^4p + b^4p^2 - c^4p^3 + 2a^2c^2p^2 - 4ab^2cp^2 = 0$; fuppofe $p = 1$, and the given equation $x^4 + qx^2 - rx + s = 0$, let the correfpondent terms of the given and refulting equations be refpectively made equal to each other, and there refult the three equations $2b^2 + 4ac = -q$, $4b(a^2 + c^2) = r$, and $a^4 - b^4 + c^4 - 2a^2c^2 + 4ab^2c = -s$; reduce thefe equations into one, fo that the unknown quantities a and c may be exterminated, and there refults the equation $4b^6 + 2qb^4 + (\frac{q^2}{4} - s)b^2 - \frac{r^2}{16} = 0$ of the formula of a cubic, from which

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the unknown quantity b may be found, which being ſubſtituted for its value (b) in the preceding equations, from the equations thence enſuing may be found the unknown quantities a and c, and conſequently the reſolution of the given biquadratic $x^4+qx^2-rx+s=0$.

From the ſame principles can be deduced different reſolutions of the above-mentioned biquadratic $x^4+qx^2-rx+s=0$.

3. I. Let $x=a\sqrt[2n]{p}+b\sqrt[2n]{p^2}$, then will the equation free from radicals be $x^{2n}-2b^n px^n-\frac{n}{1.2}.2nb^{n-1}a^2px^{n-1}-\frac{n\times \overline{n^2-1}}{1.2.3.4}\times 2nb^{n-2}a^4px^{n-2}-\frac{n\times\overline{n^2-1}\times\overline{n^2-4}}{1.2.3.4.5.6}\times 2nb^{n-3}a^6px^{n-3}-\frac{n.\overline{n^2-1}.\overline{n^2-4}.\overline{n^2-9}}{1.2.3.4.5.6.7.8}\times 2nb^{n-4}a^8px^{n-4}\ldots\ldots-\frac{n.\overline{n^2-1}\times\overline{n^2-4}\times\overline{n^2-9}.\overline{n^2-16}\ldots\overline{n^2-\overline{n-2}^2}}{1.2.3.4.5.6.7\ldots\overline{2n-2}}\times 2na^{2n-2}bpx=a^np-b^np^2$.

This equation may be deduced from the following principles. Let α, β, γ, δ, ε, &c. be the $2n$ roots of the equation $z^{2n}-1=0$, then (by Prop. XXIII. of my Meditat. Algebraicæ) the equation free from radicals will be the product of the following quantities $(x-a\alpha\sqrt[2n]{p}-b\alpha^2\sqrt[2n]{p^2})$ $(x-a\beta\sqrt[2n]{p}-b\beta^2\sqrt[2n]{p^2})$ $(x-a\gamma\sqrt[2n]{p}-b\gamma^2\sqrt[2n]{p^2})$ $(x-a\delta\sqrt[2n]{p}-b\delta^2\sqrt[2n]{p^2})$ $(x-a\varepsilon\sqrt[2n]{p}-b\varepsilon^2\sqrt[2n]{p^2})$ &c. $=0$: multiply theſe quantities into each other, and from the reſulting product, by Prob. III. of the Meditat. Algebr. eaſily can

be deduced the equation free from radicals which was to be found.

3. II. Let $x = a\sqrt[2n+1]{p} + b\sqrt[2n+1]{p^2}$, then will the correfpondent equation free from radicals be $x^{2n+1} - \overline{2n+1}\, b^n apx^n - \frac{n.\overline{n+1}}{1.2.3} \times \overline{2n+1}\, b^{n-1}a^3px^{n-1} - \frac{n \times \overline{n^2-1} \times \overline{n+2}}{1.2.3.4.5} \times \overline{2n+1}\, b^{n-2}a^5px^{n-2} - \frac{n \times \overline{n^2-1} \times \overline{n^2-4} \times \overline{n+3}}{1.2.3.4.5.6.7} \times \overline{2n+1}\, b^{n-3}a^7px^{n-3} - \frac{n.\overline{n^2-1}.\overline{n^2-4}.\overline{n^2-9}.\overline{n+4}}{1.2.3.4.5.6.7.8.9} \times \overline{2n+1}\, b^{n-4}a^9px^{n-4} - \ldots - \frac{n.\overline{n^2-1}.\overline{n^2-4}.\overline{n^2-9}.\overline{n^2-16}\ldots\overline{n^2-\overline{n-2}^2} \times \overline{2n-1}}{1.2.3.4.5.6.7.8.9\ldots\overline{2n-1}} \times \overline{2n+1}\, ba^{2n-1}px = a^{2n+1}p + b^{2n+1}p^2$.

This may be derived from the fame principles as the preceding.

3. III. In general let the equation be $x = a\sqrt[m]{p} + b\sqrt[m]{p^2}$, then will the equation free from radicals become $x^m - ma^{m-2}bpx - m.\frac{m-3}{2}a^{m-4}b^2px^2 - m.\frac{m-4}{2}.\frac{m-5}{3}a^{m-6}b^3px^3 - m.\frac{m-5}{2}.\frac{m-6}{3}.\frac{m-7}{4}a^{m-8}b^4px^4 - m.\frac{m-6}{2}.\frac{m-7}{3}.\frac{m-8}{4}.\frac{m-9}{5}a^{m-10}b^5px^6 - \&c. = a^mp \pm b^mp^2$; if m denotes an even number, it will be $-b^mp^2$, but if an odd number, it will be $+b^mp^2$.

4. I. Let n denote an odd number, and $x = a\sqrt[n]{p} + b\sqrt[n]{p^3}$, then will $x^n - p\,(na^{n-3}bx^2 + n.\frac{n-5}{2}a^{n-6}b^2x^4 + n.\frac{n-7}{2}.\frac{n-8}{3}a^{n-9}b^3x^5 + n.\frac{n-9}{2}.\frac{n-10}{3}.\frac{n-11}{4}a^{n-12}b^4x^8 + n.\frac{n-11}{2}.\frac{n-12}{3}.\frac{n-13}{4}.\frac{n-14}{5}a^{n-15}b^5x^{10} + n.\frac{n-13}{2}.\frac{n-14}{3}.\frac{n-15}{4}.\frac{n-16}{5}.\frac{n-17}{6}a^{n-18}b^6x^{12} + n.$

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$\frac{n-15}{2}.\frac{n-16}{3}.\frac{n-17}{4}.\frac{n-18}{5}.\frac{n-19}{6}.\frac{n-20}{7}a^{n-21}b^7x^{14}+\&c.)\pm p^2(na^{\frac{n-3}{2}}b^{\frac{n+1}{2}}x-\frac{1}{2^2}\times n.\frac{n-5}{2}.\frac{n-7}{3}a^{\frac{n-9}{2}}b^{\frac{n+3}{2}}x^3+\frac{1}{2^4}\times n.\frac{n-7}{2}.\frac{n-9}{3}.\frac{n-11}{4}.\frac{n-13}{5}a^{\frac{n-15}{2}}b^{\frac{n+5}{2}}x^5-\frac{1}{2^6}\times n.\frac{n-9}{2}.\frac{n-11}{3}.\frac{n-13}{4}.\frac{n-15}{5}.\frac{n-17}{6}.\frac{n-19}{7}a^{\frac{n-21}{2}}b^{\frac{n+7}{2}}x^7+\frac{1}{2^8}\times n.\frac{n-11}{2}.\frac{n-13}{3}.\frac{n-15}{4}.\frac{n-17}{5}.\frac{n-19}{6}.\frac{n-21}{7}.\frac{n-23}{8}.\frac{n-25}{9}a^{\frac{n-27}{2}}b^{\frac{n+9}{2}}x^9-\&c.)=a^np+b^np^3.$

The quantity $\pm p^2$ denotes $+p^2$ if $\frac{n-3}{4}$ is a whole number, otherwise $-p^2$.

4. II. Let n denote an even number, and x as before $=a\sqrt[n]{p}+b\sqrt[n]{p^3}$, then will $x^n-p(na^{n-3}bx^2+n.\frac{n-5}{2}a^{n-6}b^2x^4+n.\frac{n-7}{2}.\frac{n-8}{3}a^{n-9}b^3x^6+n.\frac{n-9}{2}.\frac{n-10}{3}.\frac{n-11}{4}a^{n-12}b^4x^8+n.\frac{n-11}{2}.\frac{n-12}{3}.\frac{n-13}{4}.\frac{n-14}{5}a^{n-15}b^5x^{10}+\&c.)\pm p^2(\frac{1}{2}n.\frac{n-4}{2}a^{\frac{n-6}{2}}b^{\frac{n+2}{2}}x^2-\frac{1}{2^3}\times n.\frac{n-6}{2}.\frac{n-8}{3}.\frac{n-10}{4}a^{\frac{n-12}{2}}b^{\frac{n+4}{2}}x^4+\frac{1}{2^5}\times n.\frac{n-8}{2}.\frac{n-10}{3}.\frac{n-12}{4}.\frac{n-14}{5}.\frac{n-16}{6}a^{\frac{n-18}{2}}b^{\frac{n+6}{2}}x^6-\frac{1}{2^7}\times n.\frac{n-10}{2}.\frac{n-12}{3}.\frac{n-14}{4}.\frac{n-16}{5}.\frac{n-18}{6}.\frac{n-20}{7}.\frac{n-22}{8}a^{\frac{n-24}{2}}b^{\frac{n+8}{2}}x^8+\&c.)=a^np+b^np^3\pm 2a^{\frac{n}{2}}b^{\frac{n}{2}}p^2.$

The quantities $\pm p^2$ and $\pm 2a^{\frac{n}{2}}b^{\frac{n}{2}}p^2$ denote $+p^2$, and $+2a^{\frac{n}{2}}b^{\frac{n}{2}}p^2$, if $\frac{n-2}{4}$ is a whole number, otherwise they denote $-p^2$ and $-2a^{\frac{n}{2}}b^{\frac{n}{2}}p^2$ respectively.

5. I. Let $x=a\sqrt[n]{p}+b\sqrt[n]{p^{n-1}}$, and n an odd number, then

then will $x^n - nabpx^{n-2} + n.\frac{n-3}{2}a^2b^2p^2x^{n-4} - n.\frac{n-4}{2}.\frac{n-5}{3}.$ $a^3b^3p^3x^{n-6} + n.\frac{n-5}{2}.\frac{n-6}{3}.\frac{n-7}{4}a^4b^4p^4x^{n-8} - \&c. = a^np + b^np^{n-1}$.

5. II. Let $x = a\sqrt[n]{p} + b\sqrt[n]{p^{n-1}}$, and n an even number, then will $x^n - nabpx^{n-2} + n.\frac{n-3}{2}a^2b^2p^2x^{n-4} - n.\frac{n-4}{2}.\frac{n-5}{3}a^3b^3p^3x^{n-6} + n.\frac{n-5}{2}.\frac{n-6}{3}.\frac{n-7}{4}a^4b^4p^4x^{n-8} - \&c. = a^np \mp 2a^{\frac{n}{2}}b^{\frac{n}{2}}p^{\frac{n}{2}} + b^np^{n-1}$; it will be $+ 2a^{\frac{n}{2}}b^{\frac{n}{2}}p^{\frac{n}{2}}$ if $n = 4r + 2$; but $-2a^{\frac{n}{2}}b^{\frac{n}{2}}p^{\frac{n}{2}}$ if $n = 4r$.

6. I. Let $x = a\sqrt[n]{p} + b\sqrt[n]{p^{n-2}}$, and n an odd number, which has not the number 3 for a diviſor, then will $x^n - na^2bpx^{n-3} + n.\frac{n-5}{2}a^4b^2p^2x^{n-6} - n.\frac{n-7}{2}.\frac{n-8}{3}a^6b^3p^3x^{n-9} + n.\frac{n-9}{2}.\frac{n-10}{3}.\frac{n-11}{4}a^8b^4p^4x^{n-12} - n.\frac{n-11}{2}.\frac{n-12}{3}.\frac{n-13}{4}.\frac{n-14}{5}a^{10}b^5p^5x^{n-15} + n.\frac{n-13}{2}.\frac{n \times 14}{3}.\frac{n-15}{4}.\frac{n-16}{5}.\frac{n-17}{6}a^{12}b^6p^6x^{n-6} - \&c.$ (to m terms, where m is the number either equal to, or the leaſt greater than $\frac{n}{3}$) $- ab^{\frac{n+1}{2}}p^{\frac{n-1}{2}}(nx^{\frac{n-3}{2}} + \frac{1}{4}n.\frac{n-5}{2}.\frac{n-7}{3}a^2bpx^{\frac{n-9}{2}} + \frac{1}{2^4}n.\frac{n-7}{2}.\frac{n-9}{3}.\frac{n-11}{4}.\frac{n-13}{5}a^4b^2p^2x^{\frac{n-15}{2}} + \frac{1}{2^6} \times n.\frac{n-9}{2}.\frac{n-11}{3}.\frac{n-13}{4}.\frac{n-15}{5}.\frac{n-17}{6}.\frac{n-19}{7}a^6b^3p^3x^{\frac{n-21}{2}} + \frac{1}{2^8} \times n.\frac{n-11}{2}.\frac{n-13}{3}.\frac{n-15}{4}.\frac{n-17}{5}.\frac{n-19}{6}.\frac{n-21}{7}.\frac{n-23}{8}.\frac{n-25}{9}a^8b^4p^4x^{\frac{n-27}{2}} + \&c.) = \text{A} = a^np + b^np^{n-2}$.

Let n be an odd number diviſible by 3, then will the above-

above-mentioned quantity $= \mathrm{A} = a^n p + b^n p^{n-2} + 3a^{\frac{n}{3}} b^{\frac{2n}{3}} p^{\frac{2n}{3}-1} + 3a^{\frac{2n}{3}} b^{\frac{n}{3}} p^{\frac{n}{3}}$.

6. II. Let n be an even number, not divisible by 3, then will $x^n - na^2bpx^{n-3} + n.\frac{n-5}{2} a^4b^2p^2x^{n-6} - n.\frac{n-7}{2}.\frac{n-8}{3} a^6b^3p^3x^{n-9} + n.\frac{n-9}{2}.\frac{n-10}{3}.\frac{n-11}{4} a^8b^4p^4x^{n-12} - n.\frac{n-11}{2}.\frac{n-12}{3}.\frac{n-13}{4}.\frac{n-14}{5} a^{10}b^5p^5x^{n-15} + n.\frac{n-13}{2}.\frac{n-14}{3}.\frac{n-15}{4}.\frac{n-16}{5}.\frac{n-17}{6} a^{12}b^6p^6x^{n-18} -$ &c. to m terms as before $- b^{\frac{n}{2}} p^{\frac{n}{2}-1} (2x^{\frac{n}{2}} + \frac{1}{2} n.\frac{n-4}{2} a^2bpx^{\frac{n}{2}-3} + \frac{1}{2^3} \times n.\frac{n-6}{2}.\frac{n-8}{3}.\frac{n-10}{4} a^4b^2p^2x^{\frac{n}{2}-6} + \frac{1}{2^5} \times n.\frac{n-8}{2}.\frac{n-10}{3}.\frac{n-12}{4}.\frac{n-14}{5}.\frac{n-16}{6} a^6b^3p^3x^{\frac{n}{2}-9} + \frac{1}{2^7} \times n.\frac{n-10}{2}.\frac{n-12}{3}.\frac{n-14}{4}.\frac{n-16}{5}.\frac{n-18}{6}.\frac{n-20}{7}.\frac{n-22}{8} a^8b^4p^4x^{\frac{n}{2}-12} +$ &c.$) = \mathrm{A} = a^n p - b^n p^{n-2}$.

Let n be an even number divisible by 3, then will the above-mentioned quantity $\mathrm{A} = a^n p - b^n p^{n-2} - 3a^{\frac{2n}{3}} b^{\frac{n}{3}} p^{\frac{n}{3}} + 3a^{\frac{n}{3}} b^{\frac{2n}{3}} p^{\frac{2n}{3}-1}$.

In all the preceding cases n, m and r denote whole affirmative numbers.

These equations may be deduced in the same manner as is before given in Case 3. I; or can be demonstrated by writing in the equation free from radicals for the different powers of x their values deduced from the given equation $x = a\sqrt[m]{p} + b\sqrt[m]{p^n}$.

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To render the ſolution general, it may not be improper to ſubjoin the ſubſequent.

LEMMA.

1. Let $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta$, &c. be the reſpective roots of the equation $z^n - 1 = 0$; then will $\alpha^m + \beta^m + \gamma^m + \delta^m + \varepsilon^m + \&c. = 0$, unleſs $n = m$, or n is a diviſor of m, in which caſe $\alpha^m + \beta^m + \gamma^m + \delta^m + \varepsilon^m + \&c. = n$.

2. The ſum of all quantities of the following kind $\alpha^m\beta^r + \alpha^r\beta^m + \alpha^m\gamma^r + \alpha^r\gamma^m + \beta^m\gamma^r + \beta^r\gamma^m + \alpha^m\delta^r + \&c.$ will be $= 0$; unleſs n be either equal to, or a diviſor of $m + r$, in which caſe the ſum above-mentioned will be $= -n$; except n be either equal to m or r, or a diviſor of them, in which caſe the ſum will be $n^2 - n$; but if $m = r$, then in the former caſe will the above-mentioned ſum $= -\frac{n}{2}$, and in the latter $= \frac{n^2 - n}{2}$.

3. The ſum of all quantities of this kind $\alpha^m\beta^r\gamma^s\delta^t$ &c. + $\alpha^r\beta^m\gamma^s\delta^t$ &c. + $\alpha^m\beta^r\gamma^t\delta^s$ &c. + $\alpha^r\beta^s\gamma^m\delta^t$ &c. + &c. will be $= 0$, unleſs n be either equal to $r + m + s + t$ &c. or a diviſor of it.

Let π be the number of indices m, r, s, t, &c. and n be either equal to $m + r + s + t +$ &c. or a diviſor of it, but n be neither equal to, nor a diviſor of the ſum of any two, three, four, ... $\pi - 3$, $\pi - 2$ or $\pi - 1$ of the above-mentioned

mentioned quantities; then will the sum above-mentioned $= \mp 1.2.3.4 \ldots \overline{\pi-2}.\overline{\pi-1} \times n$; where it will be +, if π be an odd number; otherwise −.

In this case, if a indices be m, b indices be r, c indices be s, d indices be t, &c. then will the above-mentioned sum $= \mp \frac{1.2.3.4 \ldots \overline{\pi-2} \times \overline{\pi-1}}{1.2.3..a \times 1.2.3..b \times 1.2.3..c \times 1.2.3..d \times \text{\&c.}} \times n$.

Let n be either equal to, or a divisor of the sum of any number ϱ (less than π) of the above-mentioned quantities m, r, s, t, &c. and consequently either equal to, or a divisor of the sum of the $(\pi-\varrho)$ remaining quantities: find the sum of all possible quantities of this kind $1.2.3..\overline{\varrho-2} \times \overline{\varrho-1} \times 1.2.3..\overline{\pi-\varrho-2} \times \overline{\pi-\varrho-1} \times n^2$, which sum call A.

Let n be either equal to, or a divisor of the sum of any number (σ) of the above-mentioned quantities m, r, s, t, &c; and also equal to, or a divisor of the sum of any number (ϱ') of the remaining quantities, and consequently it will be either equal to, or a divisor of, the sum of the $(\pi-\varrho'-\sigma)$ remaining quantities; then find the sum of all possible quantities of this sort $1.2.3..\overline{\sigma-2} \times \overline{\sigma-1} \times 1.2.3..\overline{\varrho'-2} \times \overline{\varrho'-1} \times 1.2.3..\overline{\pi-\varrho'-\sigma-2}$ $\times \overline{\pi-\varrho'-\sigma-1} \times n^3$, which sum call B.

In the same manner let n be either equal to, or a divisor of the sum of any number (τ) of the above-mentioned

tioned quantities m, r, s, t, &c; and similarly let n be either equal to, or a divisor of the sum of any number (σ') of the remaining quantities; and also let n be either equal to, or a divisor of the sum of any number (ϱ'') of the remaining quantities; then will n be either equal to, or a divisor of the sum of the $(\pi-\tau-\sigma'-\varrho'')$ remaining quantities: find the sum of all quantities of this sort $1.2.3..\overline{\tau-2}\times\overline{\tau-1}\times 1.2.3..\overline{\sigma'-2}\times\overline{\sigma'-1}\times 1.2.3..\overline{\varrho''-2}\times\overline{\varrho''-1}\times 1.2.3..\overline{\pi-\tau-\sigma'-\varrho''-2}\times\overline{\pi-\tau-\sigma'-\varrho''-1}\times n^4$, which sum call C; and so on; then will the above-mentioned sum $\alpha^m\beta^r\gamma^s\delta^t$ &c. $+\alpha^r\beta^m\gamma^s\delta^t$ &c. $+\alpha^m\beta^r\gamma^t\delta^s$ &c. $+\alpha^r\beta^s\gamma^m\delta^t$ &c. + &c. $=\mp(1.2.3..\overline{\pi-2}\times\overline{\pi-1}\times n-A+B-C+D-$ &c.) where it will be + if π be an odd number, otherwise $-$.

In this case, if a indices be m, b indices be r, c indices be s, d indices be t, &c. then will the above-mentioned sum $=\mp\frac{1.2.3..\overline{\pi-2}\times\overline{\pi-1}.n-A+B-C+D-\&c.}{1.2.3..a\times 1.2.3..b\times 1.2.3..c\times 1.2.3..d\times\&c.}$.

7. Let $\alpha, \beta, \gamma, \delta, \varepsilon$, &c. be the roots of the equation $z^n-1=0$, and the resolution be $x=a\sqrt[n]{p}+b\sqrt[n]{p^2}+c\sqrt[n]{p^3}+d\sqrt[n]{p^4}\ldots\ldots+b\sqrt[n]{p^\lambda}\ldots+k\sqrt[n]{p^\mu}\ldots.+l\sqrt[n]{p^\nu}\ldots+q\sqrt[n]{p^\xi}\ldots.+r\sqrt[n]{p^\varphi}\ldots.+s\sqrt[n]{p^{n-4}}+t\sqrt[n]{p^{n-3}}+v\sqrt[n]{p^{n-2}}+u\sqrt[n]{p^{n-1}}$; then will the different values of x be respectively $a\sqrt[n]{p}\times\alpha+b\sqrt[n]{p^2}\times\alpha^2+c\sqrt[n]{p^3}\times\alpha^3+d\sqrt[n]{p^4}\times\alpha^4\ldots.+b\sqrt[n]{p^\lambda}\times\alpha^\lambda\ldots+k\sqrt[n]{p^\mu}\times\alpha^\mu\ldots\ldots+s\sqrt[n]{p^{n-4}}\times\alpha^{n-4}+t\sqrt[n]{p^{n-3}}\times\alpha^{n-3}+v\sqrt[n]{p^{n-2}}\times\alpha^{n-2}+u\sqrt[n]{p^{n-1}}\times\alpha^{n-1}$;

$a\surd$

$a\sqrt[n]{p}\times\beta+b\sqrt[n]{p^2}\times\beta^2+c\sqrt[n]{p^3}\times\beta^3+d\sqrt[n]{p^4}\times\beta^4\ldots+h\sqrt[n]{p^\lambda}\times\beta^\lambda\ldots+k\sqrt[n]{p^\mu}\times\beta^\mu\ldots\ldots+s\sqrt[n]{p^{n-4}}\times\beta^{n-4}+t\sqrt[n]{p^{n-3}}\times\beta^{n-3}+v\sqrt[n]{p^{n-2}}\times\beta^{n-2}+u\sqrt[n]{p^{n-1}}\times\beta^{n-1}$;

$a\sqrt[n]{p}\times\gamma+b\sqrt[n]{p^2}\times\gamma^2+c\sqrt[n]{p^3}\times\gamma^3+d\sqrt[n]{p^4}\times\gamma^4\ldots+h\sqrt[n]{p^\lambda}\times\gamma^\lambda\ldots+k\sqrt[n]{p^\mu}\times\gamma^\mu\ldots\ldots+s\sqrt[n]{p^{n-4}}\times\gamma^{n-4}+t\sqrt[n]{p^{n-3}}\times\gamma^{n-3}+v\sqrt[n]{p^{n-2}}\times\gamma^{n-2}+u\sqrt[n]{p^{n-1}}\times\gamma^{n-1}$;

$a\sqrt[n]{p}\times\delta+b\sqrt[n]{p^2}\times\delta^2+c\sqrt[n]{p^3}\times\delta^3+d\sqrt[n]{p^4}\times\delta^4\ldots+h\sqrt[n]{p^\lambda}\times\delta^\lambda\ldots+k\sqrt[n]{p^\mu}\times\delta^\mu\ldots\ldots+s\sqrt[n]{p^{n-4}}\times\delta^{n-4}+t\sqrt[n]{p^{n-3}}\times\delta^{n-3}+v\sqrt[n]{p^{n-2}}\times\delta^{n-2}+u\sqrt[n]{p^{n-1}}\times\delta^{n-1}$;

&c. &c. &c. &c.

and consequently the sum of the values or roots, which is the coefficient of the second term of the equation sought, will be $a\sqrt[n]{p}\times\overline{\alpha+\beta+\gamma+\delta+\&c.}\,(o)+b\sqrt[n]{p^2}\times\overline{\alpha^2+\beta^2+\gamma^2+\delta^2+\&c.}\,(o)+c\sqrt[n]{p^3}\times\overline{\alpha^3+\beta^3+\gamma^3+\delta^3+\&c.}\,(o)+\ldots\ldots+v\sqrt[n]{p^{n-2}}\times\overline{\alpha^{n-2}+\beta^{n-2}+\gamma^{n-2}+\delta^{n-2}+\&c.}\,(o)+u\sqrt[n]{p^{n-1}}\times\overline{\alpha^{n-1}+\beta^{n-1}+\gamma^{n-1}+\delta^{n-1}+\&c.}\,(o)=0$.

The sum of the products of every two of the values or roots, which is the coefficient of the third term of the equation sought, will be $a^2\sqrt[n]{p^2}\times\overline{\alpha\beta+\alpha\gamma+\beta\gamma+\alpha\delta+\beta\delta+\gamma\delta+\&c.}\,(o)+ab\sqrt[n]{p^3}\times\overline{\alpha\beta^2+\beta\alpha^2+\alpha\gamma^2+\gamma\alpha^2+\beta\gamma^2+\gamma\beta^2+\alpha\delta^2+\delta\alpha^2+\&c.}\,(o)$, and in general all the terms will be o, unless $a\times u\times p\times\overline{\alpha\beta^{n-1}+\beta\alpha^{n-1}+\alpha\gamma^{n-1}+\gamma\alpha^{n-1}+\beta\gamma^{n-1}+}$

$\overline{\gamma\beta^{n-1}+\alpha\delta^{n-1}+\delta\alpha^{n-1}+\beta\delta^{n-1}+\delta\beta^{n-1}+\gamma\delta^{n-1}+\delta\gamma^{n-1}+\&c.}(-n)$ $+b\times v\times p\times \overline{\alpha^2\beta^{n-2}+\beta^2\alpha^{n-2}+\alpha^2\gamma^{n-2}+\gamma^2\alpha^{n-2}+\beta^2\gamma^{n-2}+\gamma^2\beta^{n-2}}$ $\overline{+\alpha^2\delta^{n-2}+\delta^2\alpha^{n-2}+\beta^2\delta^{n-2}+\delta^2\beta^{n-2}+\&c.}\ (-n)+ctp\times$ $\overline{\alpha^3\beta^{n-3}+\beta^3\alpha^{n-3}+\alpha^3\gamma^{n-3}+\gamma^3\alpha^{n-3}+\beta^3\gamma^{n-3}+\gamma^3\beta^{n-3}+\alpha^3\beta^{n-3}+}$ $\overline{\delta^3\alpha^{n-3}+\beta^3\delta^{n-3}+\delta^3\beta^{n-3}+\&c.}\ (-n)+dsp\times \overline{\alpha^4\beta^{n-4}+\beta^4\alpha^{n-4}+}$ $\overline{\alpha^4\gamma^{n-4}+\gamma^4\alpha^{n-4}\beta^4\gamma^{n-4}+\gamma^4\beta^{n-4}+\&c.}\ (-n)+\&c.=-np(au+$ $bv+ct+ds+\&c.)$

If $n=2\lambda$, then will the coefficient of b^2p be $\frac{n}{2}$, i.e. the above-mentioned coefficient will be $-np\,(au+bv+ct+ds+\ldots.+\frac{1}{2}b^2)$.

The fum of the contents of every three of the above-mentioned values or roots, which is the coefficient of the fourth term of the equation required, will be $a^3\sqrt[n]{p^3}\times\overline{\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta+\&c.}\ (o)+a^2b\sqrt[n]{p^4}\times$ $\overline{\alpha\beta\gamma^2+\alpha\gamma\beta^2+\beta\gamma\alpha^2+\alpha\beta\delta^2+\&c.}\ (o)+\&c.+a^2v\sqrt[n]{p^n}\times$ $\overline{\alpha\beta\gamma^{n-2}+\alpha\gamma\beta^{n-2}+\beta\gamma\alpha^{n-2}+\alpha\beta\delta^{n-2}+\alpha\delta\beta^{n-2}+\&c.}\ (\frac{1.2.n}{1.2})+$ $abt\sqrt[n]{p^n}\times\overline{\alpha\beta^2\gamma^{n-3}+\alpha\gamma^2\beta^{n-3}+\beta\alpha^2\gamma^{n-3}+\beta\gamma^2\alpha^{n-3}+\gamma\alpha^2\beta^{n-3}+}$ $\overline{\gamma\beta^2\alpha^{n-3}+\alpha\beta^2\delta^{n-3}\ \&c.}\ (1.2.n)+\&c$; and in general all the terms (unlefs the quantity $\sqrt[n]{p^\theta}$ contained in the term have this formula $\sqrt[n]{p^n}=p$, or $\sqrt[n]{p^{2n}}=p^2$) will be $=o$; let the general term be denoted by $blk\sqrt[n]{p^{\lambda+\mu+\nu}}\times$ $\overline{\alpha^\lambda\beta^\mu\gamma^\nu+\alpha^\lambda\beta^\nu\gamma^\mu+\alpha^\mu\beta^\lambda\gamma^\nu+\alpha^\mu\beta^\nu\gamma^\lambda+\alpha^\nu\beta^\mu\gamma^\lambda+\alpha^\nu\beta^\lambda\gamma^\mu+\alpha^\lambda\beta^\mu\delta^\nu+\&c.}$ firft let $\lambda+\mu+\nu$ neither be equal to n or $2n$, then will the

term

term above-mentioned $=0$; if it be equal to n or $2n$, then will the term be $1.2\times n\times bklp$ or $1.2nbklp^2$.

If two of the three indexes λ, μ, ν be equal to each other, then divide the above-mentioned term by 1.2; if the three indexes be equal, i. e. $\lambda=\mu=\nu$, divide it by $1.2.3$: find all quantities of this kind where $\lambda+\mu+\nu$ either is equal to n or $2n$, and add all the term from thence derived, and call the fum of them A.

The fum of the contents of every four of the values or roots above-mentioned, which is the coefficient of the fourth term of the equation required, will be $a^4\sqrt[n]{p^4}\times\overline{\alpha\beta\gamma\delta+\alpha\beta\gamma\delta+\&c.}\ (o)+a^3b\sqrt[n]{p^5}\times\overline{\alpha\beta\gamma\delta^2+\alpha\beta\gamma^2\delta+\alpha\beta^2\gamma\delta+\alpha^2\beta\gamma\delta+\&c.}\ (o)+\&c$: let $bklq\sqrt{p^{\lambda+\mu+\nu+\xi}}\times\overline{\alpha^\lambda\beta^\mu\gamma^\nu\delta^\xi+\alpha^\lambda\beta^\mu\gamma^\xi\delta^\nu+\alpha^\lambda\beta^\nu\gamma^\mu\delta^\xi+\alpha^\lambda\beta^\nu\gamma^\xi\delta^\mu+\alpha^\lambda\beta^\xi\gamma^\nu\delta^\mu+\alpha^\lambda\beta^\xi\gamma^\mu\delta^\nu+\alpha^\mu\beta^\lambda\gamma^\nu\delta^\xi}+\&c.$ denote a general term; this term will be $=0$, unlefs $\lambda+\mu+\nu+\xi$ either $=n$ or $2n$ or $3n$; in which cafe the term will be either $-1.2.3nbklqp$ or $-1.2.3nbklqp^2$ or $-1.2.3nbklqp^3$; unlefs $\lambda+\mu=\nu+\xi=n$, when the above-mentioned term will be $-(1.2.3n-n^2)\,bklqp^2$; in this cafe if $\lambda=\nu$, and confequently $\mu=\xi$, then it will be $-(1.2.3n-1.2n^2)\,bklqp^2$; but if $\lambda=\mu=\nu=\xi=\frac{n}{2}$, then will the term be $-(1.2.3n-3n^2)\,bklqp^2$.

In all thefe cafes, if two of the indexes λ, μ, ν, ξ be equal, then muft the term given above be divided by

1.2; if three, by 1.2.3; if four, by 1.2.3.4; and laſtly if two are equal to each other, and the two remaining indexes equal to each other, but not to the former two, then muſt the term aforeſaid be divided by 1.2.1.2.

Find the ſum of all the poſſible terms of this kind, which call B.

In the ſame manner from the preceding Lemma may be found the aggregates of the contents of every five, ſix, ſeven, &c. roots or values multiplied into each other, which call reſpectively C, D, E, &c; then will the equation required be $x^n * -np(au+bv+ct+ds+\&c.)x^{n-2} - Ax^{n-3}+Bx^{n-4}-Cx^{n-5}+Dx^{n-6} - \&c. = 0$.

From the ſame principles may be deduced the moſt general reduction yet known of equations to others of inferior dimenſions, *e. g.*

Let (X) $x^n+(A+a\sqrt[m]{p}+b\sqrt[m]{p^2}+c\sqrt[m]{p^3}+\ldots+s\sqrt[m]{p^{m-2}}+t\sqrt[m]{p^{m-1}})\,x^{n-1}+(B+a'\sqrt[m]{p}+b'\sqrt[m]{p^2}+\ldots+s'\sqrt[m]{p^{m-2}}+t'\sqrt[m]{p^{m-1}})\,x^{n-2}+(C+a''\sqrt[m]{p}+b''\sqrt[m]{p^2}+\&c.)\,x^{n-3}+\&c.=0$; let α, β, γ, δ, &c. be the reſpective roots of the equation $z^m-1=0$, then, from the principles before given, may be formed the different values of the equation X, which being multiplied into each other from the propoſitions beforementioned of the Meditationes Algebraicæ, may be deduced an equation of nm dimenſions free from radicals,

whose root is x, and which contains mn unknown quantities A, a, b, c, &c. B, a', b', c', &c. C, a'', b'', c'', & p: for one, two or more of these unknown quantities may be assumed any quantities whatever, and thence may be deduced equations of mn dimensions, which may be reduced to equations $x^n + (A + a\sqrt[m]{p} + b\sqrt[m]{p^2} + c\sqrt[m]{p^3} + \&c.)\, x^{n-1} + \&c. = 0$ of n dimensions.

In the same manner may be assumed equations, which involve $\sqrt[m]{p}$, $\sqrt[m]{p^2}, \ldots, \sqrt[m]{p^{m-1}}$; $\sqrt[r]{Q}$, $\sqrt[r]{Q^2}$, $\sqrt[r]{Q^3}, \ldots \sqrt[r]{Q^{r-1}}$; $\sqrt[s]{T}$, $\sqrt[s]{T^2}$, $\sqrt[s]{T^3}, \ldots, \sqrt[s]{T^{s-1}}$, &c; and from so reducing them as to exterminate the irrational quantities, may often be derived equations whose resolutions or reductions are known.

The method of transforming algebraical equations into others, whose roots bear any assignable algebraical (but not exponential) relation to the roots of a given algebraical equation first published by me in the papers sent to the Royal Society, and afterwards in the year 1760; and thirdly in my Miscellanea Analytica; and lastly in the Meditationes Algebraicæ, and since published by Mr. LE GRANGE in the Berlin Acts, is perhaps (as Mr. LE GRANGE observes) more general than Mr. HUDDE'S, or any transformation yet invented; it is very useful in the resolution of numerous problems; and further

further has this peculiar advantage over all other transformations yet invented, that it often eaſily diſcovers ſome of the firſt terms of the equation required, from which many elegant Theorems may be derived.

In the works above-mentioned, viz. Miſcell. Analyt. Medit. Algeb. &c. are given ſome problems ſerving to this transformation; the firſt of which is a ſeries, which from the coefficients of a given algebraical equation ($x^n - px^{n-1} + qx^{n-2} - \&c. = 0$) finds the ſum of any power of the roots (viz. $\alpha^m + \beta^m + \gamma^m + \delta^m +$ &c. where α, β, γ, δ, &c. denote the roots of the given equation), the law of which ſeries was publiſhed by me many years before that it was given by Mr. EULER. The third Problem often mentioned in this paper is an elegant and uſeful ſeries for finding the ſum of quantities of the following kind, viz. $\alpha^m\beta^r\gamma^s\delta^t$, &c. $+ \alpha^r\beta^m\gamma^s\delta^t$, &c. $+ \alpha^s\beta^m\gamma^r\delta^t$ &c. $+ \alpha^t\beta^m\gamma^r\delta^s$ &c. $+ \alpha^m\beta^t\gamma^r\delta^s$ &c $+$. &c.

Mr. EULER gave the following reſolution, $x = \sqrt[n]{\pi} + \sqrt[n]{\varrho} + \sqrt[n]{\sigma} + \sqrt[n]{\tau} +$ &c. where π, ϱ, σ, τ, &c. denote the roots of an equation of $n-1$ dimenſions $v^{n-1} - pv^{n-2} + qv^{n-3} -$ &c. $= 0$. It is evident, that in this caſe the equation whoſe root is x will have n^{n-1} dimenſions; for let the roots of the equation $z^n - 1 = 0$ be denoted by α, β, γ, δ, &c. then will the quantity $\sqrt[n]{\pi}$ have the n following values

lues $\alpha\sqrt[n]{\pi}$, $\beta\sqrt[n]{\pi}$, $\gamma\sqrt[n]{\pi}$, &c. and the same may be affirmed of the quantities $\sqrt[n]{\varrho}$, $\sqrt[n]{\sigma}$, $\sqrt[n]{\tau}$, &c. and consequently the quantity $\sqrt[n]{\pi}+\sqrt[n]{\varrho}$ will have $n\times n$ different values; and in the same manner the root $x=\sqrt[n]{\pi}+\sqrt[n]{\varrho}+\sqrt[n]{\sigma}+\sqrt[n]{\tau}+$ may be proved to contain $n\times n\times n\times n\times$ &c. $=n^{n-1}$ roots, and consequently in this resolution, in equations of superior dimensions, the number of independent coefficients $(n-1)$ will be very few in proportion to the number of dimensions n^{n-1}, or (if we respect its formula) n^{n-2} of the resulting equation.

Let $n=3$, and the equation resulting will rise to an equation of nine dimensions, which has the formula of a cubic; for let $x=\sqrt[3]{\pi}+\sqrt[3]{\varrho}=a$ one root, then will $\frac{-1+\sqrt{-3}}{2}a$ & $\frac{-1-\sqrt{-3}}{2}a$ be two other of the nine roots, and consequently the roots will be $\overline{x^3-a^3}\times\overline{x^3-b^3}\times\overline{x^3-c^3}=0$, which has the formula of a cubic: and in general the above-mentioned equation of n^{n-1} dimensions will, for the same reason, have the formula of an equation of n^{n-2} dimensions.

Let the resolution be $x=\sqrt[2]{\pi}+\sqrt[2]{\varrho}+\sqrt[2]{\sigma}+\sqrt[2]{\tau}+$ &c. where π, ϱ, σ, τ, &c. denote the roots of an equation $x^{n-1}-px^{n-2}+qx^{n-3}-$ &c. $=0$ of $(n-1)$ dimensions, then will the resulting equation free from radicals, whose root is x, rise to 2^{n-1} dimensions; but as every affirmative

tive has a negative root equal to it, it will have the formula of an equation of 2^{n-2} dimensions.

Let the resolution be of this formula $x = \sqrt[r\times m]{\alpha} + \sqrt[r\times m]{\beta} + \sqrt[r\times m]{\gamma} + \sqrt[r\times m]{\delta} +$ &c. if α, β, γ, δ, &c. be considered as the r power of the roots of an equation of s dimensions, then will the resulting equation, of which the resolution is given, rise only to an equation of the formula of m^{s-1} dimensions.

In the year 1762 I published some reasons, for which this method could not extend to the general resolution of algebraical equations.